Basic Theory of Interferometry

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Outline

- Diffraction from a Single (Circular) Aperture
- Young's Double Slit as Model for Astronomical Interferometers
- Fringes from Unresolved Monochromatic and Polychromatic Sources
- Fringes from Resolved Sources; Source Visibilities of Various Flavors
- Physical Interpretations and Mathematical Attributes of Fringe Visibilities
- Anecdotal Examples of Visibilities From Source Morphologies
- Wrapup

Notation

- We take our nomenclature for electromagnetic fields from Jackson. In particular, a plane parallel electromagnetic field of frequency ν propagating in free space in a direction $\hat{\mathbf{n}}$ is written as:

  $$\phi \sim A e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

  with

  $$\begin{aligned} \omega &= 2\pi\nu = 2\pi c/\lambda \\ k &= \omega/c = 2\pi\nu/c \\ \mathbf{k} &= k\hat{\mathbf{n}} \end{aligned}$$

  See Jackson Chapter 7 for more details.

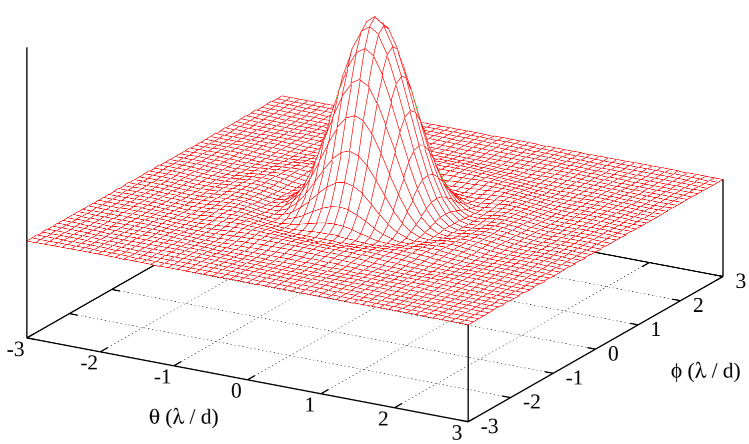
Diffraction From a Single Aperture

- Recall Diffraction From a Single Aperture. Huygens Principle Gives a Computational Mechanism For Computing the Radiation Pattern from a Single Aperture.

- Far-Field (Fraunhofer) Diffraction Pattern From a Circular Aperture:

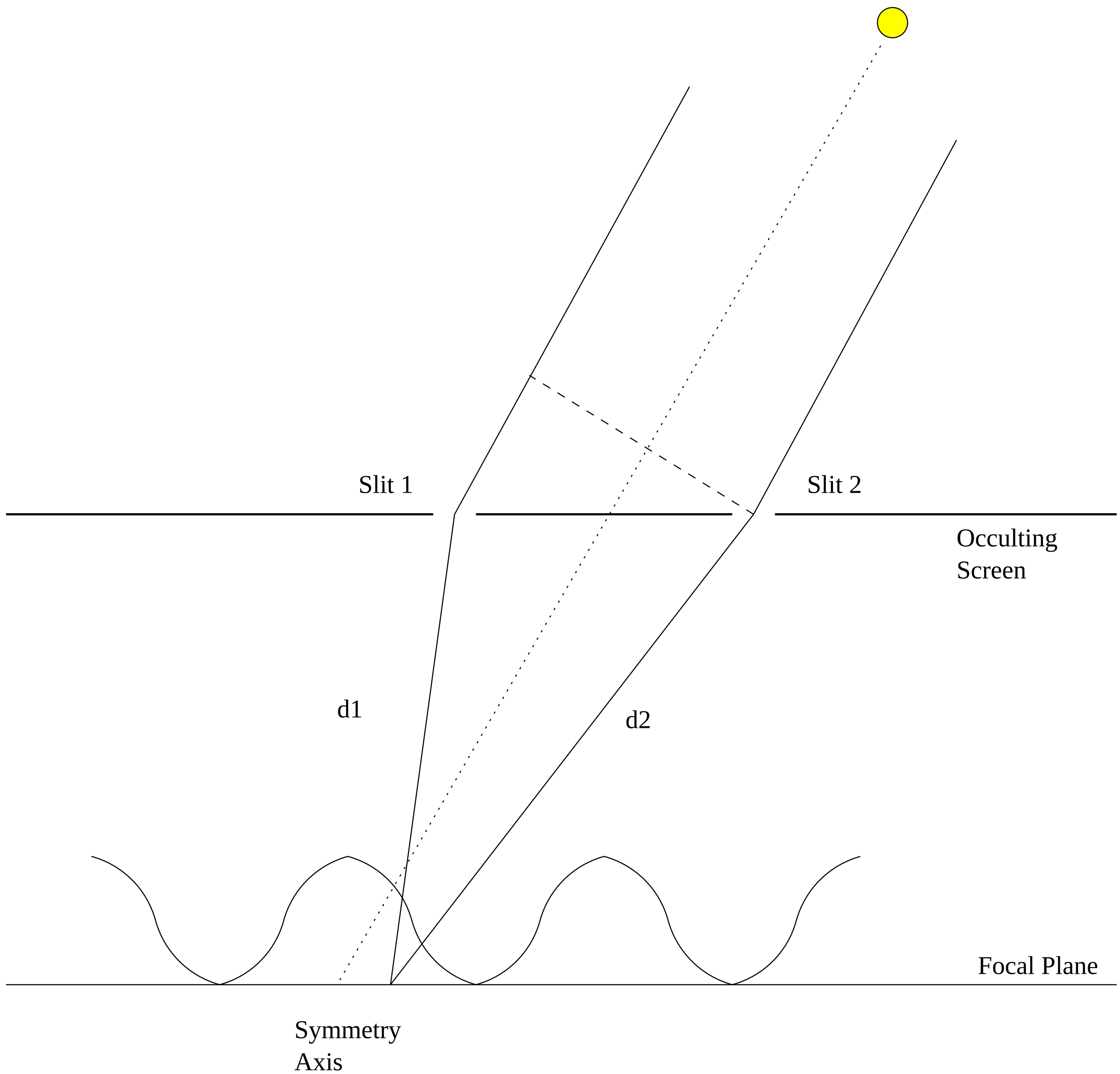
$$P(x, y) \propto \left(\frac{2J_1(\pi\theta d/\lambda)}{\pi\theta d/\lambda} \right)^2$$

- Resolution limited by characteristic angular scale λ/d

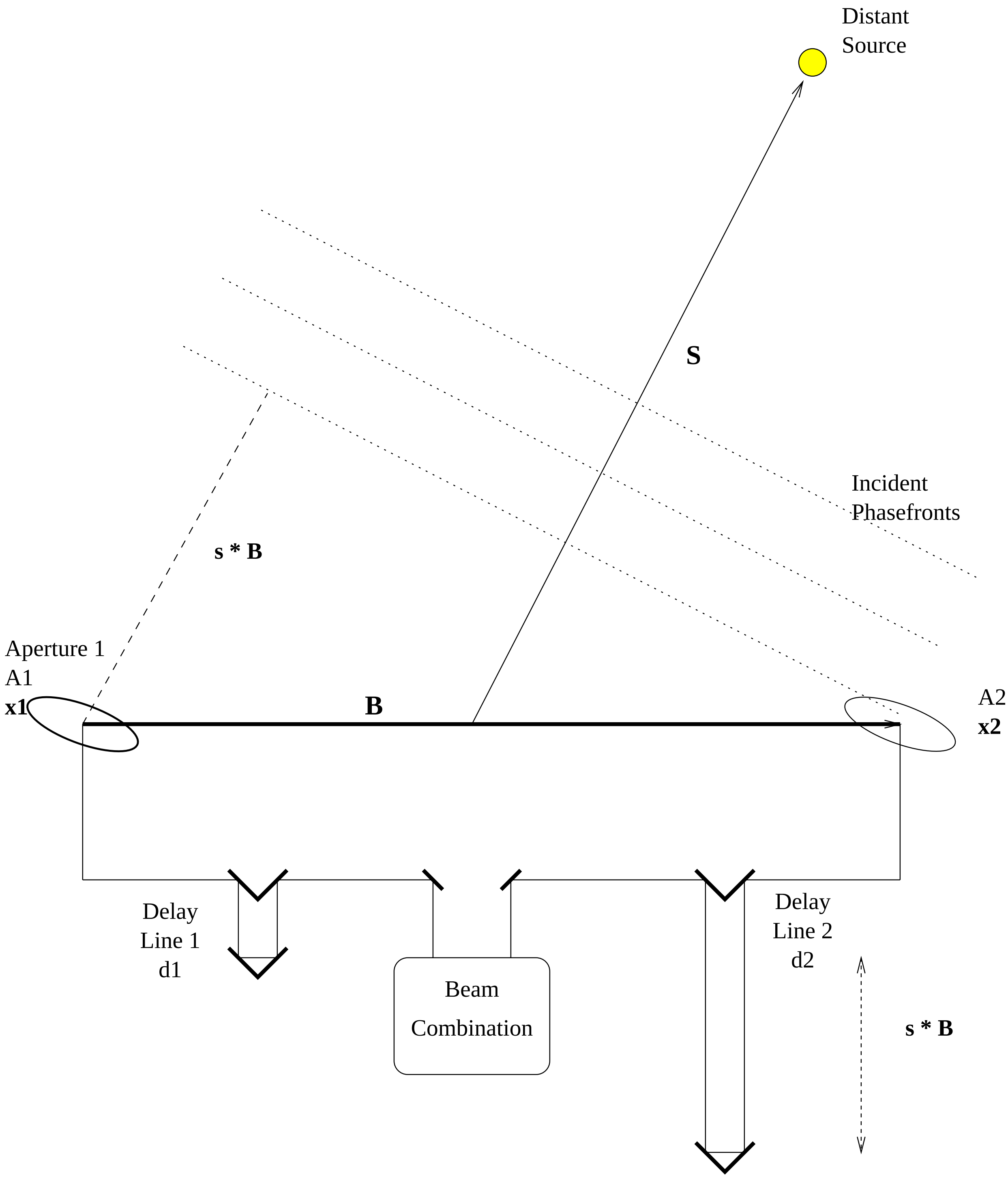


Young's Double Slit

Exploit analogy of optical interferometers and Young's double-slit experiment:



A Model Optical Interferometer



- For a *monochromatic* source, the incident optical phases are proportional to (the real part of):

$$\phi_1 \sim e^{i\mathbf{k}\cdot\mathbf{x_1}} e^{-i\omega_0 t} = e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{x_1}} e^{-i\omega_0 t} \rightarrow 1$$

  and

$$\begin{aligned}\phi_2 \sim e^{i\mathbf{k}\cdot\mathbf{x_2}} e^{-i\omega_0 t} &= e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{x_2}} e^{-i\omega_0 t} \\ &= e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{x_1}} e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{B}} e^{-i\omega_0 t} \\ &\rightarrow e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{B}}\end{aligned}$$

  absorbing a common phase factor $e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{x_1}}\, e^{-i\omega_0 t}$.

- Assuming internal pathlength delays d_1 and d_2, the fields at combination are:

$$\begin{aligned}\phi_1 &\sim e^{ik_0 d_1} \\ \phi_2 &\sim e^{ik_0 d_2}\, e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{B}} \\ \phi_{net} &\equiv \phi_1 + \phi_2 = \left(e^{ik_0 d_1} + e^{ik_0 d_2} e^{-ik_0\hat{\mathbf{s}}\cdot\mathbf{B}}\right)\end{aligned}$$

- So the time-averaged detected power is proportional* to $\phi^*_{net}\,\phi_{net}$:

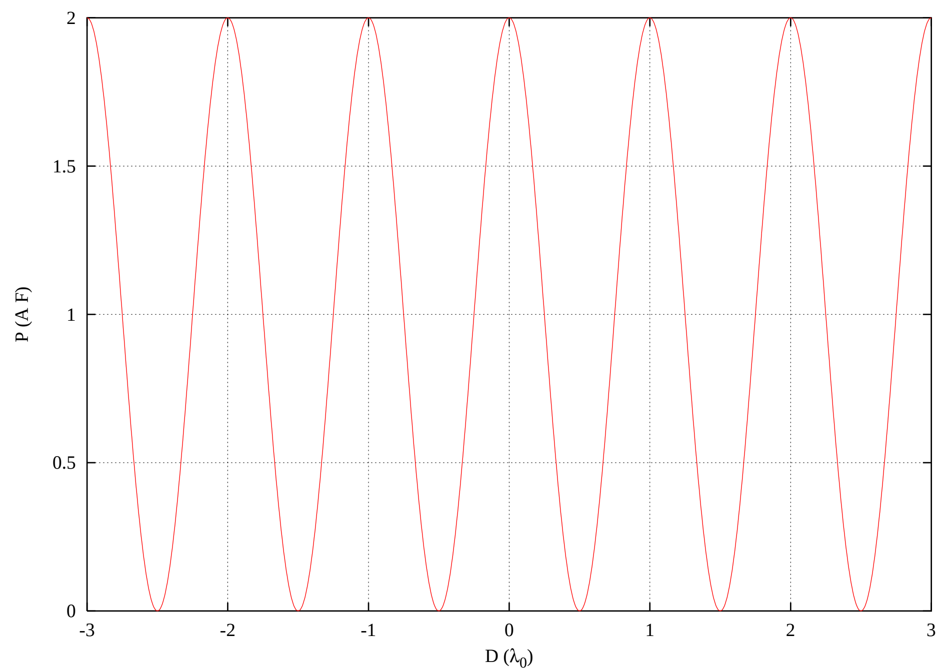
$$\begin{aligned}P \propto \phi^*_{net}\,\phi_{net} &= 2\left(1 + \cos k_0(\hat{\mathbf{s}}\cdot\mathbf{B} + d_1 - d_2)\right) \\ &= 2\left(1 + \cos k_0 D\right) \qquad (1)\end{aligned}$$

$$P = 2AF\left(1 + \cos k_0 D\right)$$

*Making some simplistic assumptions about beam combination that are incorrect in detail, but simplify the appearance of the mathematics.

Fringe Projections on the Sky...

- Eq. 1 has the form of an infinite series of power oscillations or *interference fringes*, as a function of the optical delay D, or equivalently $d_1 - d_2$.



- Because $\hat{s}$ can be interpreted as an angle on the sky with dimensions of radians, adjacent fringes projected on the sky are separated by an angle given by:

$$\Delta s = \frac{\lambda}{B} \tag{2}$$

Polychromatic Sources, Interferometers...

- The response to a spectral source flux F_ν is an incoherent sum of monochromatic responses (i.e. Eq. 1):

$$P = \int d\nu \, 2AF(\nu) \, \eta(\nu) \; [1 + \cos kD] \qquad (3)$$

- As an illustrative example take the source spectral power to be constant, F_0, over the system bandwidth, and take a specific bandwidth pattern – a "top hat" pattern with constant throughput η_0 over a band $\nu_0 \pm \Delta\nu/2$. Eq. 3 becomes:

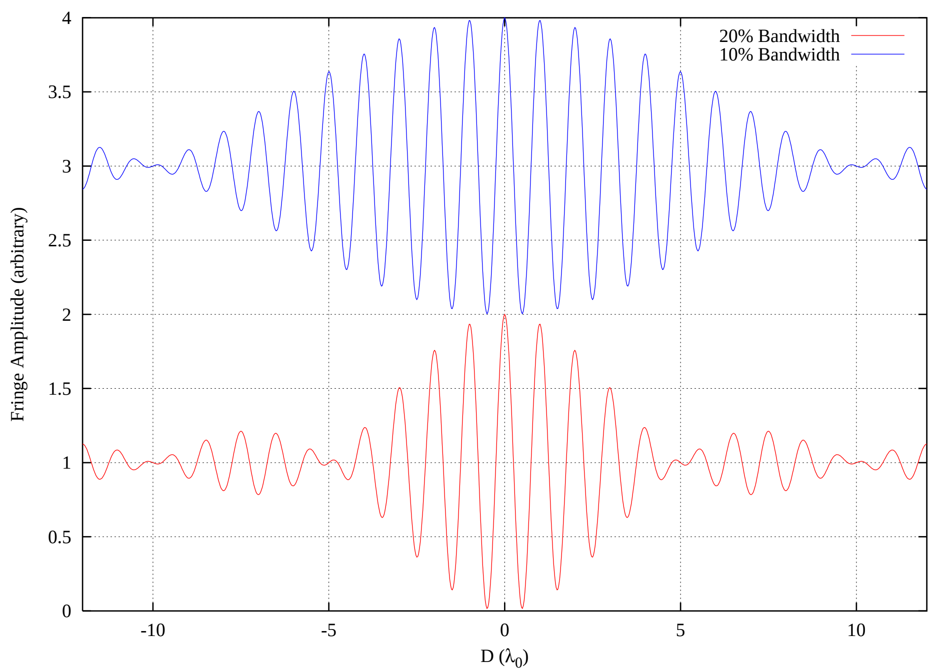
$$\begin{aligned}
P &= 2AF_{\nu-0}\eta_0 \int_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} d\nu \, (1 + \cos 2\pi\nu\tau) \\
&= 2AF_{\nu-0}\eta_0 \left[\nu + \frac{\sin 2\pi\nu\tau}{2\pi\tau}\right]_{\nu_0-\Delta\nu/2}^{\nu_0+\Delta\nu/2} \\
&= 2AF_{\nu-0}\eta_0 \Delta\nu \left[1 + \frac{\sin \pi\Delta\nu\tau}{\pi\Delta\nu\tau} \; \cos 2\pi\nu_0\tau\right] \\
&= 2AF_{\lambda-0}\eta_0 \Delta\lambda \left[1 + \frac{\sin \pi\Delta\lambda/\lambda_0^2 \; D}{\pi\Delta\lambda/\lambda_0^2 \; D} \; \cos k_0 D\right] \\
&= 2AF_{\lambda-0}\eta_0 \Delta\lambda \left[1 + \frac{\sin \pi D/\Lambda_{coh}}{\pi D/\Lambda_{coh}} \; \cos k_0 D\right] \quad (4)
\end{aligned}$$

with $\tau \equiv D/c$.

- And it is convenient to define a *coherence length* Λ_{coh}:

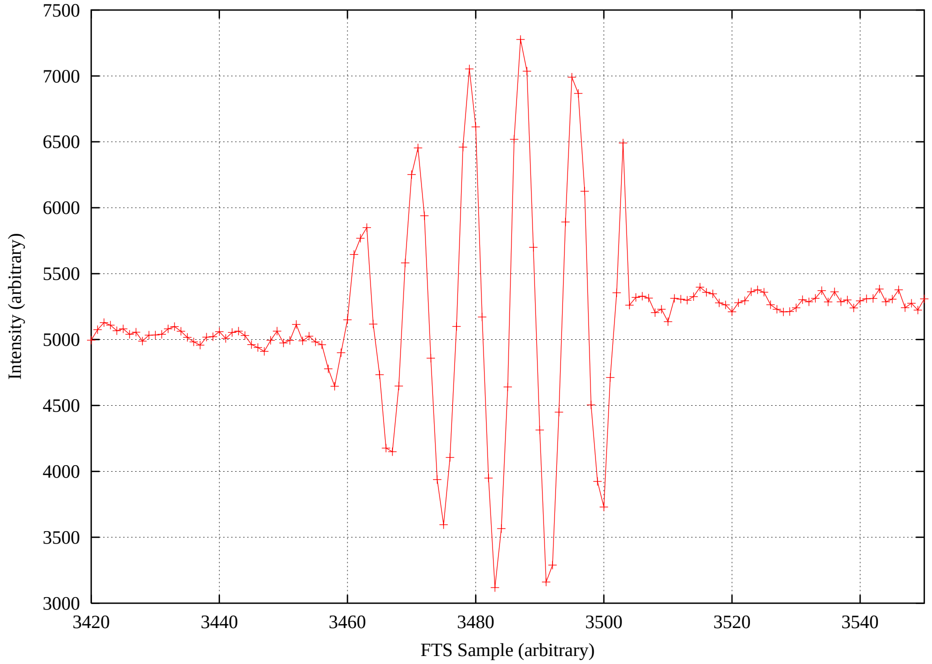
$$\Lambda_{coh} \equiv \frac{\lambda_0^2}{\Delta\lambda} \tag{5}$$

- Two illustrative examples of the oscillatory argument of Eq. 4, showing fringe patterns at 20% (red – $\Lambda_{coh} = 5\lambda_0$) and 10% (blue – $\Lambda_{coh} = 10\lambda_0$) fractional bandwidths.



- (W. Traub, next...)

- That the fringes are centered on $D = 0$ can be exploited for astrometry (Shao, Hutter, and Colavita, Wednesday morning; Johnston Friday morning).



Sample Palomar Testbed Interferometer (PTI) internal interferogram from a internal thermal source through a 20% astronomical (K-band – $\lambda_0 \sim 2.2\ \mu$m) filter. $\Lambda_{coh} \sim 5\lambda_0 \approx 11\ \mu$m.

Off-Axis Source; Phase Tracking Center...

- Conventional to *define* (control) relative delay $d_2 - d_1$ to equal (a model of) $\hat{\mathbf{s}}_0 \cdot \mathbf{B}$ – we are at maximum of fringe envelope function for source at $\hat{\mathbf{s}}_0$. In this context $\hat{\mathbf{s}}_0$ is our *phase reference* or *phase tracking center*.

- Now we ask what is response from point source at $\hat{\mathbf{s}}$ offset slightly from the reference position $\hat{\mathbf{s}}_0$:

  $$\hat{\mathbf{s}} = \hat{\mathbf{s}}_0 + \mathbf{\Delta s}$$

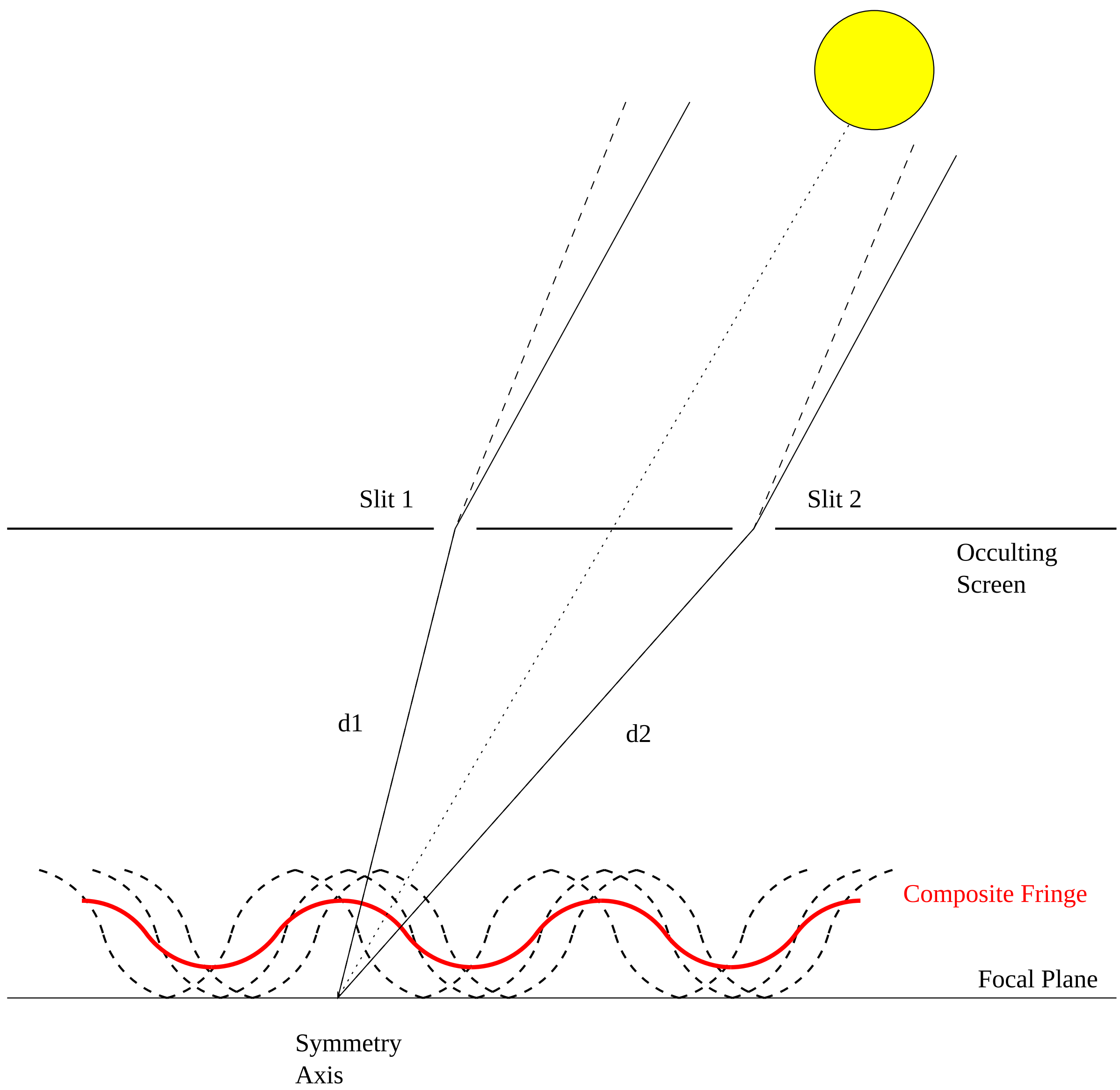
  If we write the fringe envelope function as $M(\Lambda_{coh}, \delta D)$ ($M \to 1$ in the monochromatic limit), the output power from the interferometer is:

  $$\begin{aligned} P &= 2AF\,(1 + M(\Lambda_{coh}, \Delta D)\ \cos k_0(\hat{\mathbf{s}} \cdot \mathbf{B} - \hat{\mathbf{s}}_0 \cdot \mathbf{B})) \\ &= 2AF\,(1 + M(\Lambda_{coh}, \Delta D)\ \cos k_0(\Delta\mathbf{s} \cdot \mathbf{B})) \\ &= 2AF\,(1 + M(\Lambda_{coh}, \Delta D)\ \cos k_0(\Delta D)) \qquad (6) \end{aligned}$$

  with $\Delta D \equiv \Delta\mathbf{s} \cdot \mathbf{B}$. In this construction the sky position $\hat{\mathbf{s}}_0$ as defined by the relative delay $d_2 - d_1$ *defines* the phase reference of the interference fringes on the sky.

Extended Sources

Back to the analogy of the Young's Double Slit...



Michelson defined the fringe visibility as

$$0 \leq \left[\mathcal{V}_M \equiv \frac{Max - Min}{Max + Min} \right] \leq 1$$

Extended Sources...

- Describe source intensity as function of sky position $\hat{\mathbf{s}}$ as $F(\hat{\mathbf{s}})$. Typically F has dimensions of power incident per unit area per solid angle on the sky.

- Characterize throughput (collection efficiency) of interferometer apertures as function of sky position $A(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0)$, assuming telescopes are boresighted on phase tracking center $\hat{\mathbf{s}}_0$. Convenient to take the dimensions of A as effective cross-sectional area, such that product of $A(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0)F(\hat{\mathbf{s}})\ d\Omega$ is received power differential.

- Detected power from resolved source is written as straightforward extention of Eq. 6:

$$\begin{aligned} P(\hat{\mathbf{s}}_0, \mathbf{B}) &= \int d\Omega\, A(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0) F(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0) \\ & \quad (1 + M(\Lambda_{coh}, \Delta D) \cos k(\Delta\mathbf{s} \cdot \mathbf{B})) \\ &\rightarrow \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \, (1 + \cos k(\Delta\mathbf{s} \cdot \mathbf{B})) \end{aligned} \tag{7}$$

  where I've supressed factor of 2 into magnitude of A, and dropped the envelope function assumng monochromatic source. I've also assumed that the source is spatially *incoherent* in source coordinates.

- Consider adding a small relative (to nominal tracking center) offset δ to one of the delay line arms (d_1):

$$
\begin{aligned}
P(\hat{\mathbf{s}}_0, \mathbf{B}, \delta) &= \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \left(1 + \cos k(\Delta\mathbf{s}\cdot\mathbf{B} + \delta)\right) \\
&= \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \qquad (8) \\
&\quad + \cos k\delta \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \cos k(\Delta\mathbf{s}\cdot\mathbf{B}) \\
&\quad - \sin k\delta \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \sin k(\Delta\mathbf{s}\cdot\mathbf{B})
\end{aligned}
$$

- Conventional to introduce *complex visibility* V of brightness distribution F with respect to the phase reference $\hat{\mathbf{s}}_0$ and aperture function A:

$$
V(k, \mathbf{B}) \equiv \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) e^{-ik\Delta\mathbf{s}\cdot\mathbf{B}} \qquad (9)
$$

- Using V we can write P concisely as:

$$
\begin{aligned}
P(\hat{\mathbf{s}}_0, \mathbf{B}, \delta) &= \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) \\
&\quad + Re\{V\} \cos k\delta + Im\{V\} \sin k\delta \\
&= P_0 + Re\{V e^{ik\delta}\} \qquad (10)
\end{aligned}
$$

where aperture-integrated power is
$P_0 \equiv \int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s})$

- Alternatively, Optical Interferometrists (e.g. Michelson) deal with a normalized visibility whose modulus is bounded in interval [0,1]:

$$\mathcal{V} \equiv \frac{\int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s}) e^{-ik\Delta\mathbf{s}\cdot\mathbf{B}}}{\int d\Omega\, A(\Delta\mathbf{s}) F(\Delta\mathbf{s})} \tag{11}$$

with which the detected power becomes:

$$P = P_0 \left(1 + Re\{\mathcal{V} e^{ik\delta}\}\right) \tag{12}$$

- To see why Eqs. 10 and 12 are progress, look closer at $\mathcal{V}$. To make things definite, let's take a coordinate system where $\widehat{\mathrm{s}}_0 \equiv (0,0,1)$. Making a *small field/small angle* approximation $\Delta\mathbf{s}$ is perpendicular to $\widehat{\mathrm{s}}_0$ and can be written in terms of angles α and β (units of radians):

$$\Delta\mathbf{s} \approx (\alpha, \beta, 0)$$

and the normalized visibility becomes:

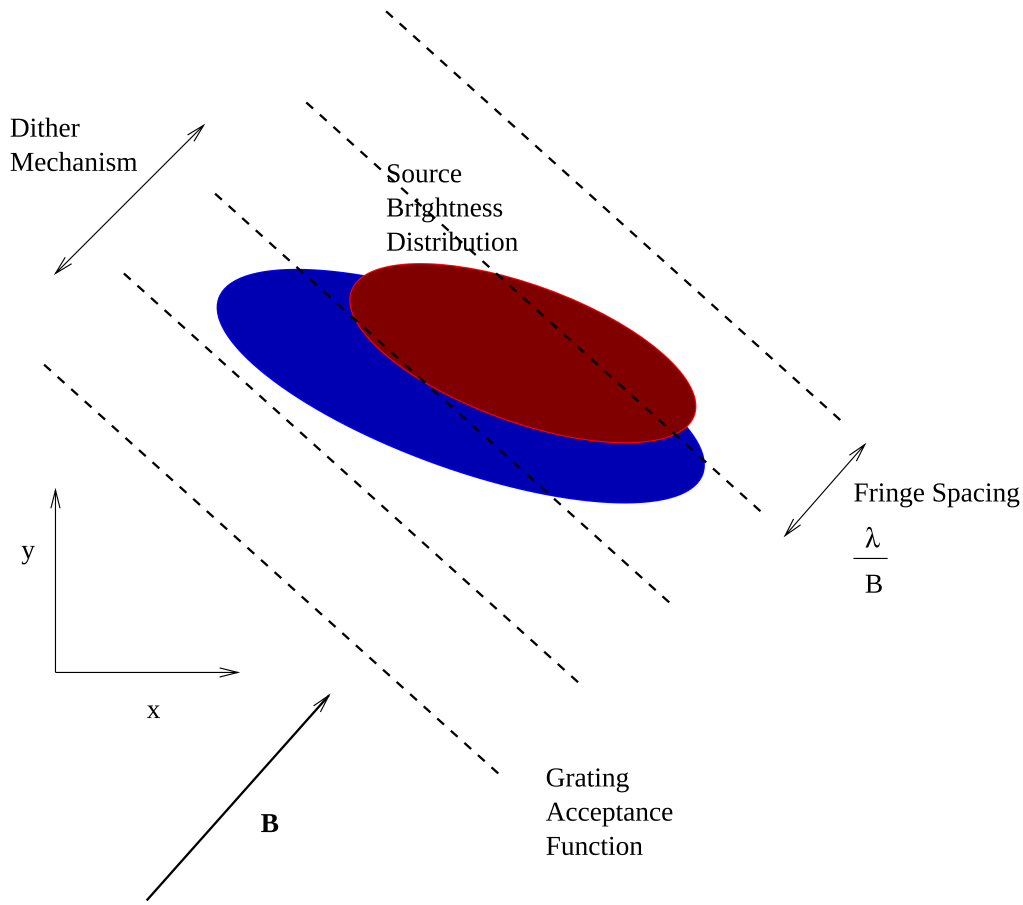
$$\begin{aligned} \mathcal{V}(k, \mathbf{B}) &= \frac{1}{P_0} \int d\alpha\, d\beta\, F(\alpha,\beta)\, e^{-ik(\alpha B_x + \beta B_y)} \\ &= \frac{1}{P_0} \int d\alpha\, d\beta\, F(\alpha,\beta)\, e^{-2\pi i(\alpha u + \beta v)} \end{aligned} \tag{13}$$

with spatial frequencies u and v defined by:

$$u \equiv \frac{B_x}{\lambda} = \frac{kB_x}{2\pi} \qquad v \equiv \frac{B_y}{\lambda} = \frac{kB_y}{2\pi} \tag{14}$$

Conventional to define u (x) and v (y) axes along the RA and Dec directions respectively.

Physical Interpretation of the Visibility



Instantaneous fringe power given by integral of brightness distribution with *cosine* kernel referenced to phase center.

(Generally complex) visibility given by integral of brightness distribution with *exponential* kernel referenced to phase center. $|\mathcal{V}| = \mathcal{V}_M$.

Different visibility components at different spatial frequencies (u, v) accessible through different baseline projections (or *possibly* wavelengths).

Mathematical Attributes of the Visibility

- Source functions are real (scalar) functions of source coordinates, so
$$V(-u,-v) = V^*(u,v)$$
...(almost like) buy one, get one free...

- Translation of source relative to phase center adds a phase to the visibility; coresponds to translation of source fringes in phase/delay space.

- Visibility for axisymmetric source is purely real (couples to the *even* part of the exponential kernel); Visibility for anti-axisymmetric source is purely imaginary (couples to the *odd* part of the exponential kernel).

- Close analogy from Q.M. with a projection of state (brightness distribution) in Hilbert space onto a component described by spatial frequency (Dirac) bra $|u,v>$.

The Canonical Imaging Procedure (CIP)...

The fact that Eq. 13 has the appearance of the familier Fourier transform suggests:

- If one can devise a scheme to measure $\mathcal{V}(u,v)$ ($V(u,v)$)...

- Measure discrete values $\mathcal{V}_j(u_j,v_j)$.

- Estimate sky brightness distribution from inverse (discrete) FT of (discrete) visibility values:

$$\begin{aligned} \mathcal{F}(\alpha,\beta) &\propto \int du\,dv\,\mathcal{V}(u,v)\,e^{2\pi i(\alpha u+\beta v)} \\ \mathcal{F}_d(\alpha,\beta) &\propto \sum_j \mathcal{V}_j(u_j,v_j)\,e^{2\pi i(\alpha u_j+\beta v_j)} \end{aligned} \tag{15}$$

- $\mathcal{F}_d(\alpha,\beta)$ is called the *dirty* image (map); it represents the true brightness distribution convolved with synthesized beam *point spread function*:

$$\mathcal{F}_d(\alpha,\beta) = \mathcal{F}(\alpha,\beta) * p(\alpha,\beta)$$

  with

$$p(\alpha,\beta) = \int du\,dv\,S(u,v)\,e^{2\pi i(u\alpha+v\beta)}$$

$$S(u,v) \equiv \sum_i \delta(u-u_i)\,\delta(v-v_i)$$

- There exist methods for estimating $\mathcal{F}$ given $\mathcal{F}_d$ (J. Monier, J. Armstrong, Wednesday afternoon)...

Visibility of Unresolved Point Source(s)

- Normalized brightness $\mathcal{F} = \delta(\alpha - \alpha_0)\,\delta(\beta - \beta_0)$

- Normalized visibility of the point source is:

$$\begin{aligned}\mathcal{V}(u,v) &= \int d\alpha\, d\beta\, \delta(\alpha - \alpha_0)\,\delta(\beta - \beta_0)\, e^{-2\pi i(\alpha u + \beta v)} \\ &= e^{-2\pi i(\alpha_0 u + \beta_0 v)} \qquad (16)\end{aligned}$$

  i.e. a pure phase for all u and v, with modulus one. This is what we **mean** by **unresolved** in the visibility language.

- Multiple (similar) sources through linear superposition:

$$\mathcal{V}(u,v) \sim \frac{1}{n}\sum_j e^{-2\pi i(\alpha_j u + \beta_j v)}$$

- Two (equal brightness) sources an interesting case:

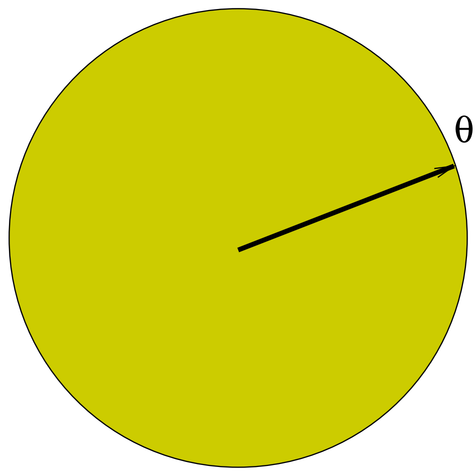
$$\begin{aligned}\mathcal{V}(u,v) &= \frac{1}{2}\left(e^{-2\pi i(\alpha_0 u + \beta_0 v)} + e^{-2\pi i(\alpha_1 u + \beta_1 v)}\right) \\ &= \frac{1}{2} e^{-2\pi i(\alpha_0 u + \beta_0 v)}(1 + e^{-2\pi i(\Delta\alpha u + \Delta\beta v)}) \\ |\mathcal{V}(u,v)|^2 &= \frac{1}{2}(1 + \cos 2\pi(\Delta\alpha u + \Delta\beta v))\end{aligned}$$

- The two sources are "unresolved" by the interferometer ($|\mathcal{V}(u,v)|^2 \sim 1$) when $(\Delta\alpha u + \Delta\beta v) \to \Delta\theta\ B/\lambda \approx 0$ (duh).

  The sources become "resolved" ($|\mathcal{V}(u,v)|^2 \sim 1/2$) when $\Delta\theta \sim \lambda/4B$.

Visibility of the Uniform Disk

Stars emit a good fraction of their light in the visible and infrared parts of the spectrum, and are (reasonably) approximated by uniform disks (van Belle and Hajian, Friday morning):

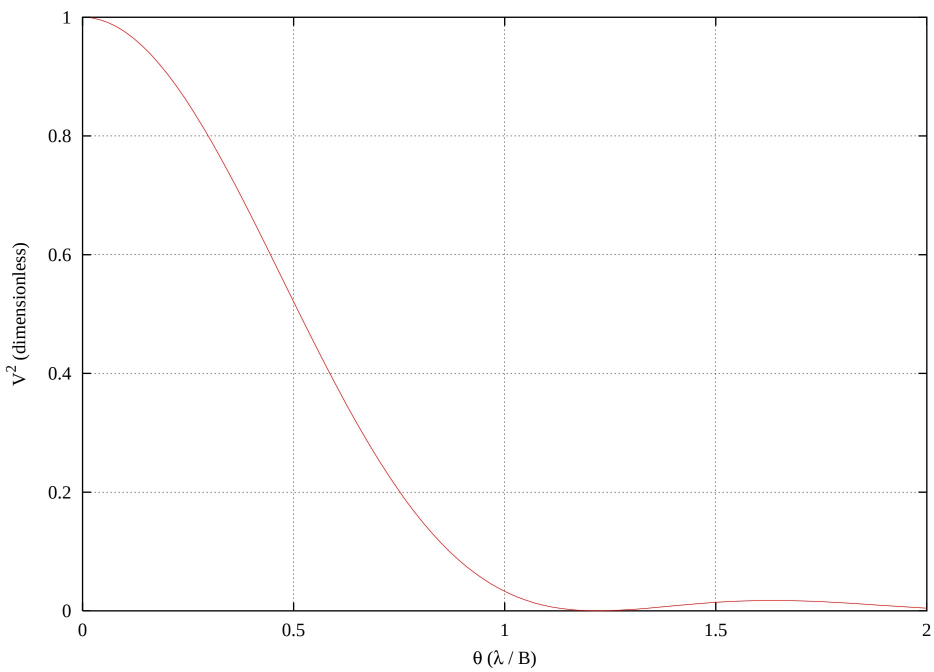


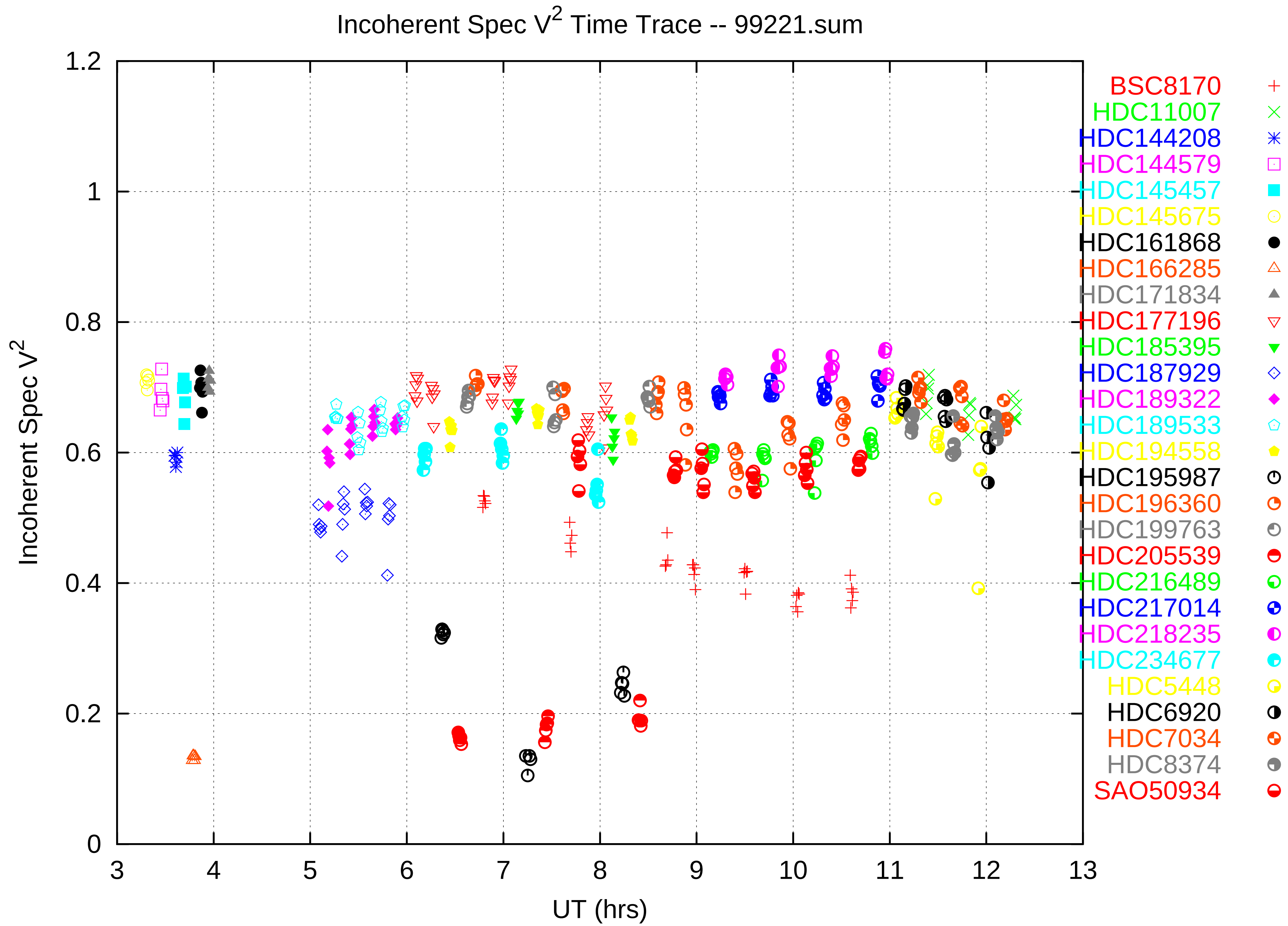
- Uniform disk diameter θ, normalized brightness $\mathcal{F} = \frac{4}{\pi\theta^2}$.

- Normalized visibility given by:

$$
\begin{aligned}
\mathcal{V}(u,v) &= \int d\alpha\, d\beta\, \frac{4}{\pi\theta^2}\, e^{-2\pi i(u\alpha+v\beta)} \\
&= \frac{4}{\pi\theta^2}\int_0^{\frac{\theta}{2}} d\rho\,\rho \int_0^{2\pi} d\phi\, e^{-2\pi i\rho v_r \cos\phi} \\
&= \frac{8}{\pi\theta^2}\int_0^{\frac{\theta}{2}} d\rho\,\rho\; J_0(2\pi v_r \rho) \\
&= \frac{2J_1(\pi v_r \theta)}{\pi v_r \theta} = \frac{2J_1(\pi\; B/\lambda\;\theta)}{\pi\; B/\lambda\;\theta}
\end{aligned}
$$

- Compare to diffraction from a single aperture:
  - Functional form is the same – visibility and Fraunhofer diffraction use the same integral kernel (Green's function).
  - Characteristic angular scales given by λ / B in both cases.
  - Uniform disk becomes "resolved" ($\mathcal{V}^2 \sim 1/2$) when $D \sim \lambda/2B$ – "just" like filled aperture.





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Wrapup

What Should You Take Away From This Talk:

- Interferometers Aren't Particularly Mysterious (at Least in *Theory*).

- Interferometers Provide Similar Resolving Power to Conventional Filled Apertures of Same Size. We (Presumably) Trade Fabrication Cost For Added Complexity in Data Interpretation.

- Observables Are Fringes in Delay Domain. Fringe Amplitudes – *Visibilities* – of Source Morphologies are Predictable With Simple Theory. The Interferometer "Samples (A Component of) the Fourier Transform of the Source Brightness Distribution".

- For *A Priori* Unknown Morphologies Measurement of Source Visibility vs. Spatial Frequency Can Be Inverted to Infer Parent Brightness Distribution. (*Monnier and Armstrong Wednesday afternoon*)

- Even When Ample Spatial Frequency Coverage is Inaccessible, Or Only Certain Components of the Source Visibility is Available, Theory Can Still Be Exploited to Infer Interesting Physics For Simple Source Morphology Models. (*Dyck/Lawson this afternoon; van Belle, Hartkopf, Hajian, and Danchi Friday morning*)

- **All We Have to Do Is Measure Visibilities!** (*Traub this morning; Colavita and Mozurkewich Wednesday afternoon*)

References

- Born and Wolf, *Principles of Optics*
- Jackson, *Classical Electrodynamics*
- NRAO Summer School, *Synthesis Imaging in Radio Astronomy*
- Thompson, Moran, and Swenson, *Interferometry and Synthesis on Radio Astronomy*